Announcements

1) Problems to turn in Thursday; #'s 2, 6

2) Exam 2 EC - turn in exam!

3) Survey - names!

<u>Recall:</u> We're finding the area between the x-axis and the graph of $y = x^2$ from $x = 0$ to $x = 1$.

We used rectangles to approximate the area and came up with

$$\frac{1}{n^3} \cdot \left(1 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2\right)$$

for n rectangles of equal base.

Then the area should be

$$\lim_{n\to\infty} \frac{1}{n^3}\Big(\underbrace{1+2^2+3^2+\cdots+(n-1)^2+n^2}\Big)$$

We need a good formula for this sum in order to take the limit

Try something easier first:

Is there a formula for

$1+2+3+\cdots+(n-1)+n$?

Picture

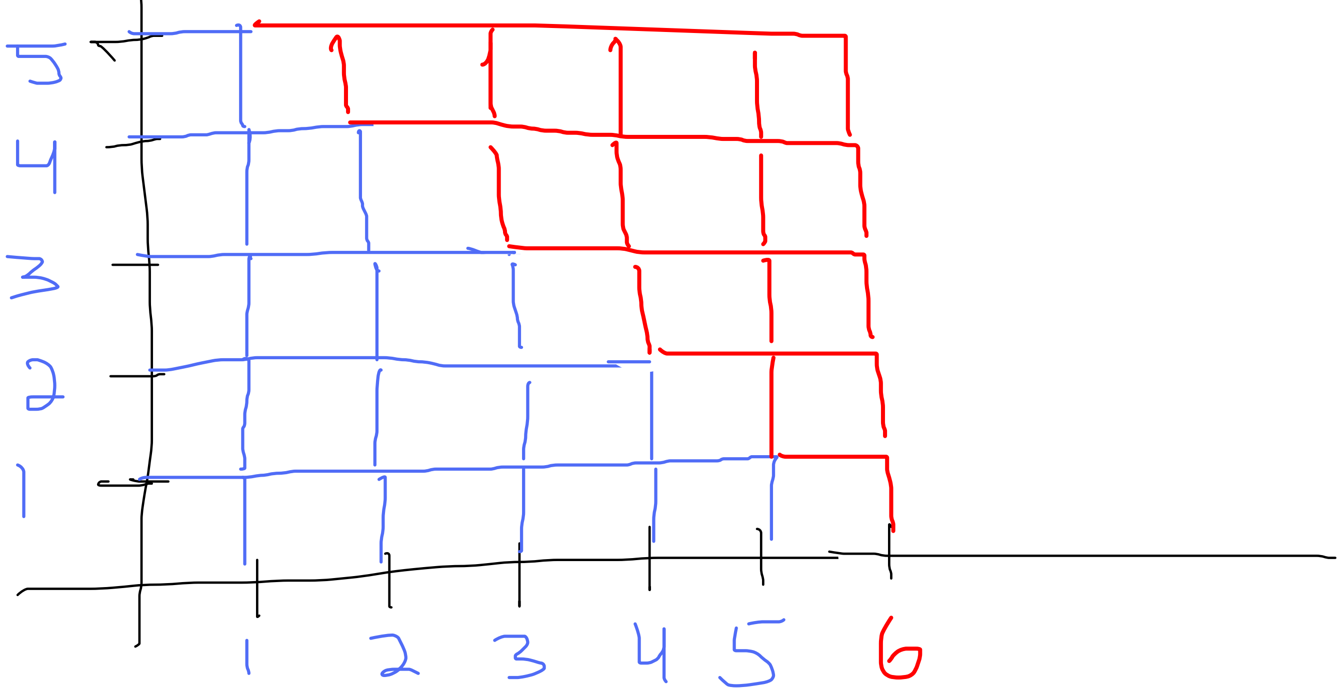
$n = 5$

$1+2+3+4+5$

= area of blue squares

= area of red squares

$= \frac{1}{2}$ (area of rectangle formed by all boxes)



For $n=5$, the area of the large rectangle is

$5 \cdot 6 = 30$, so

$$1+2+3+4+5 = \frac{1}{2} \cdot 30 = 15.$$

The same holds true in general:

$$1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}$$

For $1+2^2+\cdots+(n-1)^2+n^2$, you'd use cubes instead of squares to get

$$\boxed{1+2^2+3^2+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}}$$

So we get

$$\text{Area} = \lim_{n\to\infty} \frac{1}{n^3}\left(1+2^2+3^2+ - + (n-1)^2+n^2\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{2n^3+3n^2+1}{6n^3}$$

$$= \lim_{n\to\infty} \frac{2n^3}{6n^3} \quad \text{(highest powers)}$$

$$= \boxed{\frac{1}{3}}$$

General Continuous Functions?

Definition:

Let f be a function defined on the interval $[a,b]$. We say f is **integrable** on $[a,b]$ if

$$\lim_{n\to\infty} \frac{b-a}{n}\left(f\left(a+\frac{b-a}{n}\right)+f\left(a+\frac{2(b-a)}{n}\right)+f\left(a+\frac{3(b-a)}{n}\right)+\cdots+f\left(a+\frac{(n-1)(b-a)}{n}\right)+f(b)\right)$$

exists.

We denote by $\boxed{\int_a^b f(x)\,dx}$ the value of

$$\lim_{n\to\infty} \frac{b-a}{n}\left(f\left(a+\frac{b-a}{n}\right)+f\left(a+\frac{2(b-a)}{n}\right)+f\left(a+\frac{3(b-a)}{n}\right)+\cdots+f\left(a+\frac{(n-1)(b-a)}{n}\right)+f(b)\right)$$

provided the limit exists and call it the definite integral of f on $[a,b]$.

Note: The "dx" in the integral is a placeholder that tells you what the dependent variable is that you're using to calculate the integral.

<u>Definition:</u> The area between the graph of f and the x-axis is given by

$$\boxed{A = \int_a^b |f(x)|\,dx}$$

again provided the limit exists

Difference between Area and Integral

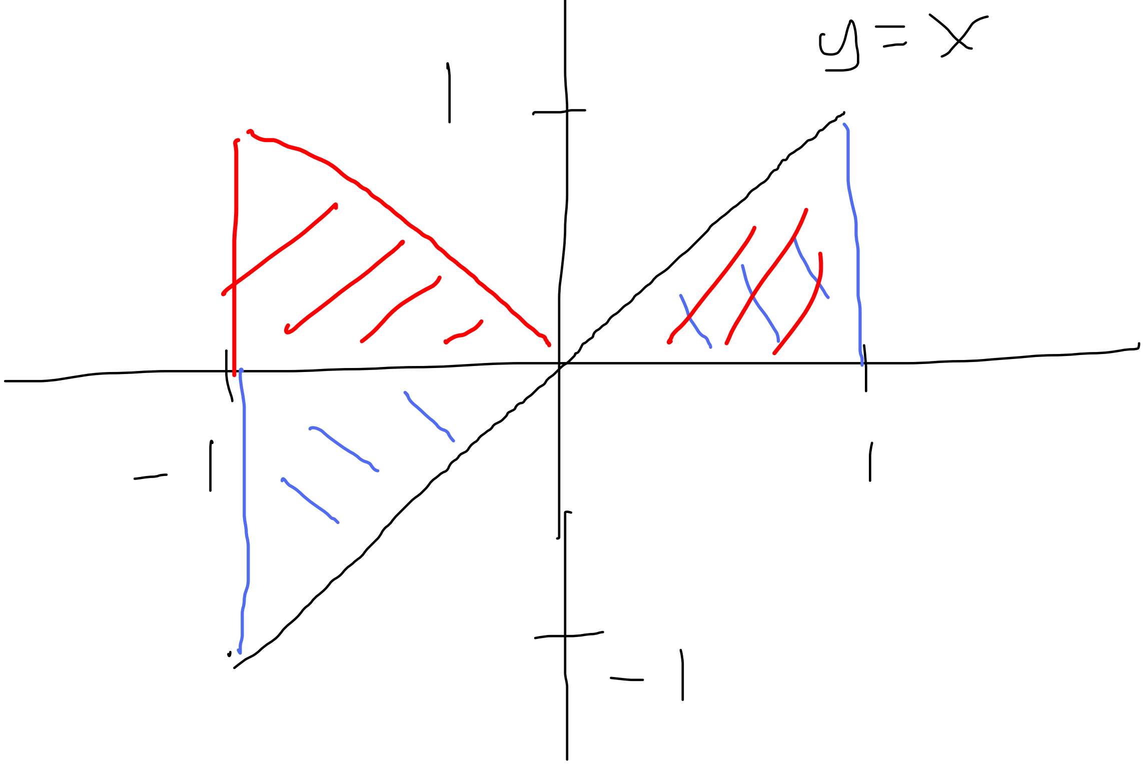
If $f(x) \geq 0$ for all x with $a \leq x \leq b$, then

$$\int_a^b |f(x)|\,dx = \int_a^b f(x)\,dx$$

so there is no difference in this case. In general, though, there can be a huge difference.

Example 1: $\int_{-1}^{1} x\,dx$ vs. $\int_{-1}^{1} |x|\,dx$

Picture:



The area $\int_{-1}^{1} |x| \, dx$ is just the area of two of the triangles, which is $\boxed{1}$

The integral $\int_{-1}^{1} x \, dx$ is the difference of areas of two triangles, which is $\boxed{0}$

Properties of Integrals:

Suppose $\int_a^b f(x)\,dx$, $\int_a^b g(x)\,dx$

both exist. Then

1) $\int_a^b (f(x)+g(x))\,dx$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2) $\int_a^b (f(x)-g(x))\,dx$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

3) If c is any constant,

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$$

4) If $f(x) \leq g(x)$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

5) If $a \le c \le b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The Fundamental Theorem of Calculus

Observation. $\int_0^1 x^2\,dx$ sucks

$\int_0^1 x^3\,dx$ sucks more

Fundamental Theorem of Calculus

(2 parts)

Let f be a continuous function on $[a,b]$

Part One

If $a \le x \le b$, define

a function

$$\boxed{g(x) = \int_a^x f(t)\,dt}$$

Then g is continuous on $[a, b]$,

differentiable on (a, b), and

for all x with $a \le x \le b$,

$$\boxed{g'(x) = f(x)}$$

Part Two

If h is any antiderivative of f,

$$\boxed{\int_a^b f(x)\,dx = h(b) - h(a)}$$

Example 2: $\int_0^1 x^2\,dx$

An antiderivative for $f(x) = x^2$

is $h(x) = \frac{x^3}{3}$

The by the second part of the

fundamental theorem,

$$\int_0^1 x^2\,dx = h(1) - h(0) = \boxed{\frac{1}{3}}$$